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Comparing the Fuzzy Hazard Rate Function of Three Parameters of Weibull Distribution

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This paper researches the fuzzy hazard rate function of three parameters of Weibull distribution, by Moments Estimators (MOE), Maximum Likelihood Estimators (MLE), and the Regression Method. The comparison was done by simulation, using different sample sizes (n = 40, 60, 80) and initial values of (b, c, δ) with b = scale, c =shape, δ = location parameters. The parameters were estimated by these three different methods (MLE, MOM, PEC(Regression) and the values of ti is generated from C.D.F, using inverse transformation. Then we took a set of five values of ti for application and estimation, to choose the best estimators that gave the smallest mean square error, as explained in the Tables.

Key words: *Weibull Distribution, Maximum Likelihood Estimator (MLE), Moments Estimator(MOM), Percentiles', Three parameter Weibull, (b: scale parameter), (c: shape parameter) and,(δ: location parameter) , n= 40 ,60 ,80 (sample size) , kˇ= 0.3 , 0.6 (fuzzy factor).*

Introduction

Weibull distribution has received much interest in reliability theory and it is used to describe real phenomena and modelling distribution of the breaking strength of materials. Also, it is used to fit tree diameters data, where this data plays an important role in stand modelling. This distribution was introduced by Bailey and Dell (1973) as a model for diameter distribution and has been applied extensively (Gupta and Ma, 1996) in forestry since it has the ability to describe a wide range of Uniondale distributions. Little and Kilkki et al (1989) as well as Cran (1988) worked on moment estimators for three parameter of Weibull distribution. While Tsionas (2000) introduced posterior analysis, prediction and reliability in three parameter of Weibull distribution . Weibull distribution was used to describe real phenomena and modelling distribution of the breaking strength of materials as indicated by Johnson et al (2018)and also by Murthy et al (2004) . It was also used in Tsionas' (2000) work on estimating Bayesian and prediction and reliability in three parameters of Weibull (Cran, 2017). Tian (2015)refers to coefficient of variation. . A group of researchers introduced the transmuted exponential of Weibull distribution with application where this distribution is flexible and useful for analysing positive data that has a bathtub shaped hazard rate function (Cordeiro et al., 2014). The obtained new distribution can be used for modelling positive data in various fields such as physical and biological sciences; reliability theory hydrology,; medicine and survival analysis; and engineering (Kilkkip and Paivinenr, 1989). (Felix and Chukwudi, 2014) introduced a comparison between different methods for estimating parameters of Weibull, they used mean square error as a measure for comparison. Mahdi and Arjunk (2013) Gupta gave more attention for estimating the reliability function for three parameters of Weibull and variation and maximum likelihood estimation. Also, they referred to Cran (1988), where he estimated three parameters by moments of Weibull (Barreto – Souza et al., 2010).

Theory

We know that Weibull distribution arises from exponential distribution, when we assume the (pth) power of failure time of a random variable has an exponential distribution if:

y= T^r is r.v $\sim negative\ exponential\ with\ mean\ \theta\ then$

$$g(y,\theta) = \frac{1}{\theta}\ e^{-\frac{y}{\theta}} \qquad y,\theta > 0 \qquad (1)$$

Then the failure time (T) is r.v have Weibull distribution (WEb) and its failure rate proportional to some power t, (Kundu and Raqab, 2009)

$h(t) = \alpha\ t^k$

$h(t) = c\ t^k$

$Let: k = p - 1 \quad and\ c = \frac{p}{\theta}$

Then T is r.v ~ WEb

$$f(t) = \frac{p}{\theta} \quad t^{p-1} \quad e^{-\frac{t^p}{\theta}} \qquad p,\theta > 0 \qquad (2)$$

We can prove that

$\mu'_r = E(t^r)$ for f(t) in equation (1)

Equal $\mu'_r = E(t^r)$

$$= \theta^{\frac{r}{p}}\ \Gamma(1+\frac{r}{p}) \qquad (3)$$

When r =1 → $E(T) = \theta^{\frac{1}{p}}\ \Gamma(1+\frac{r}{p})$

And variance of (T) is

$$\sigma^2{}_T = E(T^2) + (ET)^2$$

$$\sigma^2{}_T = \theta^{\frac{2}{p}} \left[\Gamma \left(1 + \frac{2}{p}\right) - \Gamma^2 \ (1 + \frac{1}{p}) \right] \qquad (4)$$

Here our research deals with three parameters of Weibull which was introduced by Petrosian (1994), when the failure happen after a certain time (i.e.) $t > \delta \quad , \delta < t < \infty$:

$$F_T(t, b, c, \delta) = \frac{c}{b} \left(\frac{t-\delta}{b}\right)^{c-1} \ e^{-\left(\frac{t-\delta}{b}\right)^c} \qquad (5)$$

Were δ: is location parameter
b: is scale parameter
c: shape parameter

Corresponding to equation (5), the cumulative distribution function (CDF) is:

$$F_T(t, b, c, \delta) = 1 - \ e^{-\left(\frac{t-\delta}{b}\right)^c} \qquad t \geq \delta \qquad (6)$$

And the reliability function is

$$R_T(t) = pr \ (T > t)$$

$$R_T(t) = e^{-\left(\frac{t-\delta}{b}\right)^c} \qquad (7)$$

We also proved that the r th moments formula of three parameters of Weibull is:

$$\mu^{\times}{}_r = \sum_{i=0}^{r} \binom{r}{i} \ \mathrm{b^r} \ \ \delta^{r-i} \ \ \Gamma \left(\frac{r}{c} + 1\right) \qquad (8)$$

We can use equation (8) and Solving equation (9):

$$\mu^{\times}{}_r = \frac{\sum_{i=1}^{n} ti^r}{n} \qquad for\ r = 1, 2, 3 \qquad (9)$$

to obtain $\hat{b}_{mom}$, $\hat{c}_{mom}$, $\hat{\delta}_{mom}$

Maximum Likelihood Method Estimation

Let $(t_1, t_2, \dots, t_n)$ be ar.s from p.d.f in equation (5), then:

$$L = \prod_{i=1}^{n} f\ (t_i, b, c, \delta)$$

$$L = c^n \; b^{-n} \prod_{i=1}^{n} \left(\frac{t_i - \delta}{b}\right)^{c-1} \; e^{-\sum_{i=1}^{n}\left(\frac{t-\delta}{b}\right)^{c}}$$

$$L = c^n \; b^{-n(c-1)} \prod_{i=1}^{n} (t_i - \delta)^{c-1} \; e^{-\sum_{i=1}^{n}\left(\frac{t-\delta}{b}\right)^{c}} \qquad (10)$$

Taking logarithmic for equation (10) we obtained:

$$\log L = n \log c - n\,(c-1)\log b + (c-1)\sum_{i=1}^{n} \log(t_i - \delta) - \; b^{-c}\sum_{i=1}^{n} (t_i - \delta)^c \qquad (11)$$

Then from

$$\frac{\partial \log L}{\partial c} = \frac{n}{c} - n\log b + \sum_{i=1}^{n} \log(t_i - \delta) - \left(b^{-c}\sum_{i=1}^{n} (t_i - \delta)^c \; (1)\log(t_i - \delta) + \sum_{i=1}^{n} (t_i - \delta)^c\,(-c)\; b^{-c-1} \right)$$

Put $\frac{\partial \log L}{\partial c} = 0$

$$\frac{n}{\hat{c}} - n\log b + \sum_{i=1}^{n} \log(t_i - \delta) - \; b^{-c}\sum_{i=1}^{n}(t_i - \delta)^c \; \log(t_i - \delta) - c \;\; b^{-c-1}\sum_{i=1}^{n} (t_i - \delta)^c = 0 \qquad (12)$$

equation (12) solved numerically to find $\hat{c}_{MLE}$

Now:

$$\frac{\partial \log L}{\partial b} = \left[\frac{-nc}{b} + c \;\; b^{-c-1} \;\; \sum_{i=1}^{n}(t_i - \delta)^c\} * \;\; b^{c+1}\right] \div c$$

$$n * \; b^c = \sum_{i=1}^{n}(t_i - \delta)^c$$

$$b^c = \frac{\sum_{i=1}^{n}(t_i - \delta)^c}{n} \quad ; \text{ then}$$

$$\hat{b}_{MLE} = \sqrt[\hat{c}]{\frac{\sum_{i=1}^{n}(t_i - \delta)^{\hat{c}}}{n}} \qquad (13)$$

And:

$$\frac{\partial \log L}{\partial \delta} = (c-1)\sum_{i=1}^{n} \frac{-1}{(t_i - \delta)} - \; b^{-c}\sum_{i=1}^{n} c \;\; (t_i - \delta)^{c-1}\,(-1)$$

$$= (c-1)\sum_{i=1}^{n} (t_i - \delta)^{-1} + c \;\; b^{-c}\sum_{i=1}^{n} (t_i - \delta)^{c-1} \quad = 0 \qquad (14)$$

Solving equation (14) numerically to find ($\hat{\delta}_{MLE}$)

Third Method Regression Estimation

$$F\ (t) = 1 - e^{-\left(\frac{t-\delta}{b}\right)^c}$$

from

$$e^{-\left(\frac{t-\delta}{b}\right)^c} = 1 - F(ti)$$

$$e^{-\left(\frac{t-\delta}{b}\right)^c} = wi \qquad 0 \le wi \le 1 \qquad (15)$$

Taking Logarithm for equation (15); then

$$-\left(\frac{t-\delta}{b}\right)^c = \log wi$$

we assume $\log wi = yi$, then

$$(-yi)^{\frac{1}{c}} = \left(\frac{ti-\delta}{b}\right)$$

$$\hat{ti} = \left(\ b\ (-yi)^{\frac{1}{c}} + \ \delta\ \right)$$

Then estimation by regression obtained from minimising the total sum of squares between t_i and $\hat{ti}$:

$$T = \sum_{i=1}^{n} \left[t_{(i)} - \left(b\ \ (-1)^{\frac{1}{c}}\ \ (yi)^{\frac{1}{c}} + \delta \right) \right]^2$$

$$T = \sum_{i=1}^{n} \left[t_{(i)} - b\ \ (-yi)^{\frac{1}{c}} - \delta \right]^2 \qquad (16)$$

From $\frac{\partial T}{\partial b} = 0$ we find:

$$\hat{b}_{(Reg)} = \frac{\left(\sum_{i=1}^{n} t_{(i)}\ (-yi)^{\frac{1}{c}} - \hat{\delta}\ \sum_{i=1}^{n}\ (-yi)^{\frac{1}{c}}\right)}{\left(\sum_{i=1}^{n}\ (yi)^{\frac{2}{c}}\right)} \qquad (17)$$

Also, from $\frac{\partial T}{\partial c} = 0$ we obtained the nonlinear equation:

$$\sum_{i=1}^{n} (-1)^{\frac{1}{c}}\ t_{(i)}\, yi\ \log(\frac{1}{yi}) - b \sum_{i=1}^{n} (-yi)^{\frac{2}{c}}\ \log(\frac{1}{yi}) + \hat{\delta} \sum_{i=1}^{n} (-1)^{\frac{1}{c}}\, yi\ \log\ (yi^{-1}) = 0 \qquad (18)$$

From equation (18) solved numerically we found $\hat{c}_{(Reg)}$

$$\frac{\partial T}{\partial \delta} = 2 \sum_{i=1}^{n}\ (t_{(i)} - b - (yi)^{\frac{1}{c}} - \delta)\ (1) = 0$$

$$=\sum_{i=1}^{n} t_{(i)} - n\hat{b} - \sum_{i=1}^{n} (yi)^{\frac{1}{\hat{c}}} = n\hat{\delta} \qquad \div n$$

$$\hat{\delta}_{MLE} = t - \hat{b} - \frac{\sum_{i=1}^{n} (yi)^{\frac{1}{c}}}{n} \tag{19}$$

Simulation Procedure

To find $(\hat{b},\ \hat{c}, \hat{\delta})$ we apply simulation procedure, taking sample size $(n = 40, 60, 80)$, and using inverse transformation of (CDF) as:

$$U_i = F\ (t_i\ , \delta, b\ , c)$$

$$U_i = 1 - e^{-\left(\frac{t_i - \delta}{b}\right)^c} \qquad t \le \delta$$

$$e^{-\left(\frac{t_i - \delta}{b}\right)^c} = 1 - U_i$$

$$\log(1 - U_i) = -\left(\frac{t_i - \delta}{b}\right)^c \qquad 0 \le U_i \le 1$$

$$-\log(1 - U_i) = \left(\frac{t_i - \delta}{b}\right)^c$$

Let $zi = -\log(1 - U_i)$ and $wi = (zi)^{\frac{1}{c}}$

Then $zi = \left(\frac{t_i - \delta}{b}\right)^c$

$when\ wi = (zi)^{\frac{1}{c}}$

Then $(zi)^{\frac{1}{c}} = \frac{t_i - \delta}{b}$

$$wi = \frac{t_i - \delta}{b}$$

$$b\ wi = t_i - \delta$$

$$t_i = b\ wi + \delta \qquad t_i \ge \delta$$

Using generated values (t_i) for (n = 40 ,60, 80) we can find moment estimators from equation (9) and maximum likelihood from equations (12,13,14). Finally we applied regression analysis of (t_i) at n = 40, 60, 80 to obtain the least square estimator for $(\hat{b}\ , \hat{c}, \hat{\delta})$; from equation (17,18,19).

The results of estimation were compared using the statistical measure of mean square error (MSE) and as explained in the following table, we notice some values of $\hat{h}\ (t_i)$ are greater than (1), this indicates that the value of p.d.f at some set is greater that the value of $R_T(t)$, at this set .

Table 1: includes Table 1.1, Table 1.2, Table1.3 and Table1.4

Table 1.1: Estimator fuzzy hazard rate when (c=3, b=2.8, ʒ=0.5, $\check{k}$ =0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.1584	0.0994	0.0151	0.0379	0.0007	0.0041	0.0045	mom
	1.1500	0.2548	0.0999	0.0091	0.0379	0.0048	0.0121	0.0122	mom
	1.4500	0.3513	0.1004	0.0332	0.1137	0.0126	0.0202	0.0215	mom
	1.7500	0.4477	0.1009	0.0574	0.1895	0.0241	0.0305	0.0334	mom
	2.0500	0.5441	0.1014	0.0816	0.2653	0.0392	0.0428	0.0478	mom
60	0.8500	0.1584	0.1019	0.0175	0.0202	0.0006	0.0040	0.0038	mom
	1.1500	0.2548	0.1024	0.0094	0.0202	0.0046	0.0120	0.0110	mom
	1.4500	0.3513	0.1029	0.0363	0.0606	0.0123	0.0198	0.0169	mom
	1.7500	0.4477	0.1034	0.0632	0.1010	0.0237	0.0296	0.0240	mom
	2.0500	0.5441	0.1040	0.0901	0.1415	0.0387	0.0412	0.0324	reg
80	0.8500	0.1584	0.1347	0.0422	0.0079	0.0001	0.0027	0.0029	mom
	1.1500	0.2548	0.1357	0.0001	0.0079	0.0028	0.0130	0.0094	mom
	1.4500	0.3513	0.1366	0.0419	0.0236	0.0092	0.0191	0.0113	mom
	1.7500	0.4477	0.1376	0.0840	0.0393	0.0192	0.0265	0.0133	reg
	2.0500	0.5441	0.1386	0.1260	0.0551	0.0329	0.0350	0.0156	reg

Table 1.2: Estimator fuzzy hazard rate when (c=3, b=2.8, ʒ=0.5, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.3551	0.0878	0.0005	0.0069	0.0143	0.0251	0.0251	mom
	1.1500	0.5480	0.0886	0.0109	0.1315	0.0422	0.0577	0.0568	mom
	1.4500	0.7408	0.0893	0.0224	0.2562	0.0849	0.1032	0.1012	mom
	1.7500	0.9337	0.0901	0.0338	0.3808	0.1423	0.1619	0.1585	mom
	2.0500	1.1265	0.0909	0.0453	0.5054	0.2145	0.2338	0.2284	mom
60	0.8500	0.3551	0.1209	0.0002	0.0008	0.0110	0.0251	0.0250	mom
	1.1500	0.5480	0.1224	0.0446	0.0151	0.0362	0.0507	0.0545	mom
	1.4500	0.7408	0.1239	0.0894	0.0293	0.0761	0.0849	0.0953	mom
	1.7500	0.9337	0.1254	0.1342	0.0436	0.1307	0.1278	0.1474	Mle
	2.0500	1.1265	0.1269	0.1790	0.0578	0.1998	0.1795	0.2108	Mle
80	0.8500	0.3551	0.1491	0.0018	0.0014	0.0085	0.0250	0.0242	mom
	1.1500	0.5480	0.1514	0.0485	0.0260	0.0314	0.0499	0.0347	mom
	1.4500	0.7408	0.1538	0.0988	0.0506	0.0689	0.0824	0.0470	reg
	1.7500	0.9337	0.1561	0.1491	0.0752	0.1209	0.1231	0.0611	reg
	2.0500	1.1265	0.1585	0.1994	0.0999	0.1874	0.1719	0.0772	reg

Table 1.3: Estimator fuzzy hazard rate when (c=3, b=3.2, ʒ=0.5, $\check{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.0926	0.0892	0.0991	0.0669	0.0002	0.0001	0.0004	mle
	1.1500	0.1770	0.0896	0.0647	0.0423	0.0015	0.0025	0.0036	mom
	1.4500	0.2613	0.0900	0.0303	0.0178	0.0107	0.0059	0.0119	mle
	1.7500	0.3457	0.0904	0.0041	0.0068	0.0230	0.0233	0.0130	reg
	2.0500	0.4301	0.0908	0.0385	0.0314	0.0307	0.0230	0.0318	mle
60	0.8500	0.0926	0.0909	0.0581	0.0945	0.0001	0.0002	0.0003	mom
	1.1500	0.1770	0.0913	0.0383	0.0598	0.0015	0.0023	0.0027	mom
	1.4500	0.2613	0.0918	0.0184	0.0251	0.0102	0.0058	0.0112	mle
	1.7500	0.3457	0.0922	0.0014	0.0096	0.0231	0.0129	0.0226	mle
	2.0500	0.4301	0.0926	0.0213	0.0443	0.0304	0.0228	0.0298	mle
80	0.8500	0.0926	0.1092	0.0529	0.0781	0.0001	0.0003	0.0001	mom
	1.1500	0.1770	0.1099	0.0345	0.0494	0.0023	0.0041	0.0009	reg
	1.4500	0.2613	0.1105	0.0161	0.0207	0.0120	0.0045	0.0107	mle
	1.7500	0.3457	0.1112	0.0023	0.0080	0.0110	0.0236	0.0220	mom
	2.0500	0.4301	0.1118	0.0207	0.0367	0.0287	0.0335	0.0203	reg

Table 1.4: Estimator fuzzy hazard rate when (c=3, b=3.2, ʒ=0.5, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.3316	0.0857	0.0009	0.0023	0.0121	0.0219	0.0219	mom
	1.1500	0.5004	0.0865	0.0615	0.0443	0.0343	0.0385	0.0470	mom
	1.4500	0.6691	0.0872	0.1240	0.0863	0.0677	0.0594	0.0816	mle
	1.7500	0.8379	0.0880	0.1865	0.1284	0.1125	0.0849	0.1257	mle
	2.0500	1.0066	0.0887	0.2489	0.1704	0.1685	0.1793	0.1148	reg
60	0.8500	0.3316	0.1106	0.0018	0.0008	0.0098	0.0218	0.0217	mom
	1.1500	0.5004	0.1119	0.1755	0.0156	0.0302	0.0211	0.0416	mle
	1.4500	0.6691	0.1132	0.3492	0.0303	0.0618	0.0679	0.0205	reg
	1.7500	0.8379	0.1145	0.5228	0.0450	0.1047	0.0199	0.1007	mle
	2.0500	1.0066	0.1157	0.6965	0.0598	0.1587	0.0192	0.1399	mle
80	0.8500	0.3316	0.1266	0.0009	0.0009	0.0084	0.0215	0.0215	mom
	1.1500	0.5004	0.1283	0.0172	0.0177	0.0277	0.0167	0.0410	mle
	1.4500	0.6691	0.1300	0.0353	0.0345	0.0581	0.0543	0.0204	reg
	1.7500	0.8379	0.1317	0.0534	0.0513	0.0997	0.0183	0.1007	mle
	2.0500	1.0066	0.1334	0.0715	0.0681	0.1525	0.0149	0.1262	mle

Table 2: Includes Table 2.1, Table 2.2, Table 2.3 and Table 2.4

Table 2.1: Estimator fuzzy hazard rate when (c=5, b=2.8, ʒ=0.5, $\check{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
	0.8500	0.1543	0.1414	0.0083	0.0017	0.0043	0.0001	0.0047	mle
	1.1500	0.2949	0.1424	0.0054	0.0011	0.0168	0.0047	0.0173	mle
40	1.4500	0.4355	0.1435	0.0025	0.0005	0.0171	0.0375	0.0379	mom
	1.7500	0.5762	0.1445	0.0004	0.0002	0.0664	0.0663	0.0373	reg
	2.0500	0.7168	0.1456	0.0033	0.0008	0.0653	0.1018	0.1027	mom
	0.8500	0.1543	0.1777	0.0115	0.7475e-3	0.0001	0.0041	0.0047	mom
	1.1500	0.2949	0.1794	0.0075	0.4729e-4	0.0165	0.0027	0.0170	mle
60	1.4500	0.4355	0.1812	0.0035	0.1983e-3	0.0373	0.0129	0.0379	mle
	1.7500	0.5762	0.1829	0.0005	0.0763e-3	0.0309	0.0663	0.0660	mom
	2.0500	0.7168	0.1846	0.0045	0.3509e-3	0.1015	0.0566	0.1025	mle
	0.8500	0.1543	0.2129	0.0093	0.2285e-3	0.0047	0.0040	0.0001	reg
	1.1500	0.2949	0.2155	0.0062	0.1446e-3	0.0013	0.0162	0.0167	mom
80	1.4500	0.4355	0.2180	0.0030	0.0606e-3	0.0370	0.0095	0.0379	mle
	1.7500	0.5762	0.2205	0.0001	0.0233e-3	0.0253	0.0662	0.0658	mom
	2.0500	0.7168	0.2230	0.0033	0.1073e-3	0.1012	0.0488	0.1025	mle

Table 2.2: Estimator fuzzy hazard rate when (c=5, b=2.8, ʒ=0.5, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
	0.8500	0.5527	0.1383	0.0001	0.0002	0.0344	0.0611	0.0611	Mom
	1.1500	0.8340	0.1402	0.0061	0.0040	0.1371	0.0963	0.1378	Mle
40	1.4500	1.1152	0.1422	0.0123	0.0078	0.2433	0.1894	0.2453	mle
	1.7500	1.3965	0.1441	0.0185	0.0117	0.3137	0.3798	0.3835	Mom
	2.0500	1.6777	0.1460	0.0247	0.0155	0.5526	0.5465	0.4692	reg
	0.8500	0.5527	0.1861	0.0002	0.0100	0.0611	0.0269	0.0611	Mle
	1.1500	0.8340	0.1896	0.0066	0.1895	0.1369	0.0830	0.1362	Mle
60	1.4500	1.1152	0.1932	0.0131	0.3690	0.1700	0.2429	0.2444	Mom
	1.7500	1.3965	0.1968	0.0196	0.5485	0.3792	0.2878	0.3833	Mle
	2.0500	1.6777	0.2004	0.0260	0.7280	0.4465	0.5456	0.5525	Mom
	0.8500	0.5527	0.1823	0.0002	0.0000	0.0610	0.0263	0.0611	Mle
	1.1500	0.8340	0.1858	0.0059	0.0009	0.0822	0.1354	0.1360	Mom
80	1.4500	1.1152	0.1893	0.0121	0.0018	0.2431	0.1687	0.2434	Mle
	1.7500	1.3965	0.1928	0.0183	0.0027	0.3832	0.3790	0.2866	reg
	2.0500	1.6777	0.1963	0.0245	0.0035	0.5451	0.4390	0.5506	Mle

Table 2.3: Estimator fuzzy hazard rate when (c=5, b=3.2, ʒ=0.5, $\check{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.1543	0.1414	0.0083	0.0017	0.0047	0.0043	0.0001	reg
	1.1500	0.2949	0.1424	0.0054	0.0011	0.0047	0.0168	0.0173	Mom
	1.4500	0.4355	0.1435	0.0025	0.0005	0.0375	0.0171	0.0379	Mle
	1.7500	0.5762	0.1445	0.0004	0.0002	0.0373	0.0663	0.0664	Mom
	2.0500	0.7168	0.1456	0.0033	0.0008	0.1018	0.0653	0.1025	Mle
60	0.8500	0.1543	0.1777	0.0115	0.7475e-3	0.0041	0.0001	0.0045	Mle
	1.1500	0.2949	0.1794	0.0075	0.4729e-3	0.0165	0.0027	0.0170	Mle
	1.4500	0.4355	0.1812	0.0035	0.1983e-3	0.0373	0.0129	0.0375	Mle
	1.7500	0.5762	0.1829	0.0005	0.0763e-3	0.0309	0.0663	0.0664	Mom
	2.0500	0.7168	0.1846	0.0045	0.3509e-3	0.1020	0.1015	0.0566	reg
80	0.8500	0.1543	0.1881	0.0074	0.5662e-3	0.0002	0.0038	0.0042	Mom
	1.1500	0.2949	0.1900	0.0049	0.3582e-3	0.0022	0.0162	0.0165	Mom
	1.4500	0.4355	0.1919	0.0023	0.1502e-3	0.0373	0.0119	0.0371	Mle
	1.7500	0.5762	0.1938	0.0002	0.0578e-3	0.0292	0.0661	0.0664	Mom
	2.0500	0.7168	0.1957	0.0027	0.2658e-3	0.1010	0.0543	0.1018	Mle

Table 2.4: Estimator fuzzy hazard rate when (c=5, b=3.2, ʒ=0.5, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.5527	0.1392	0.0001	0.0002	0.0342	0.0611	0.0611	Mom
	1.1500	0.8340	0.1412	0.0085	0.0040	0.0960	0.1363	0.1391	Mom
	1.4500	1.1152	0.1432	0.0172	0.0077	0.2411	0.1890	0.2486	Mle
	1.7500	1.3965	0.1452	0.0259	0.0115	0.3757	0.3132	0.3898	Mle
	2.0500	1.6777	0.1471	0.0346	0.0152	0.5400	0.4686	0.5528	Mle
60	0.8500	0.5527	0.1866	0.0002	0.0076	0.0611	0.0610	0.0268	reg
	1.1500	0.8340	0.1902	0.0071	0.1444	0.0829	0.1361	0.1378	Mom
	1.4500	1.1152	0.1937	0.0139	0.2812	0.2408	0.1698	0.2453	Mle
	1.7500	1.3965	0.1973	0.0207	0.4180	0.3753	0.2876	0.3832	Mle
	2.0500	1.6777	0.2009	0.0275	0.5548	0.4362	0.5322	0.5526	Mom
80	0.8500	0.5527	0.1889	0.0006	0.0000	0.0610	0.0265	0.0610	Mle
	1.1500	0.8340	0.1926	0.0103	0.0003	0.1357	0.0823	0.1365	Mle
	1.4500	1.1152	0.1964	0.0211	0.0006	0.1689	0.2394	0.2451	Mom
	1.7500	1.3965	0.2002	0.0320	0.0009	0.3830	0.3724	0.2862	reg
	2.0500	1.6777	0.2039	0.0428	0.0012	0.4344	0.5312	0.5521	Mom

Table 3: Includes Table 3.1, Table 3.2, Table 3.3 and Table3.4

Table 3.1: Estimator fuzzy hazard rate when (c=5, b=2.8, ʒ=0.8, $\check{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.1365	0.1611	0.0343	0.0016	0.0020	0.0021	0.0037	Mom
	1.1500	0.2972	0.1625	0.0222	0.0010	0.0176	0.0158	0.0036	reg
	1.4500	0.4579	0.1638	0.0102	0.0004	0.0401	0.0173	0.0419	Mle
	1.7500	0.6186	0.1652	0.0019	0.0002	0.0411	0.0761	0.0768	Mom
	2.0500	0.7793	0.1666	0.0139	0.0007	0.1172	0.0751	0.1212	Mle
60	0.8500	0.1365	0.2023	0.0476	0.7108	0.0025	0.0009	0.0036	Mle
	1.1500	0.2972	0.2046	0.0309	0.4497	0.0017	0.0151	0.0173	Mom
	1.4500	0.4579	0.2068	0.0143	0.1886	0.0419	0.0394	0.0126	reg
	1.7500	0.6186	0.2091	0.0023	0.0725	0.0760	0.0335	0.0765	Mle
	2.0500	0.7793	0.2113	0.0190	0.3336	0.1156	0.0645	0.1210	Mle
80	0.8500	0.1365	0.2423	0.0239	0.2054	0.0001	0.0016	0.0036	Mom
	1.1500	0.2972	0.2455	0.0158	0.1299	0.0005	0.0142	0.0170	Mom
	1.4500	0.4579	0.2488	0.0076	0.0545	0.0388	0.0087	0.0419	Mle
	1.7500	0.6186	0.2521	0.0006	0.0210	0.0763	0.0758	0.0269	reg
	2.0500	0.7793	0.2554	0.0088	0.0964	0.1204	0.1152	0.0549	Mle

Table 3.2: Estimator fuzzy hazard rate when (c=5, b=2.8, ʒ=0.8, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.4005	0.1539	0.2330	0.0030	0.0122	0.0056	0.0321	Mle
	1.1500	0.7219	0.1564	0.0932	0.0011	0.0640	0.0791	0.1042	Mom
	1.4500	1.0434	0.1589	0.0466	0.0007	0.1565	0.1987	0.2177	Mom
	1.7500	1.3648	0.1614	0.1864	0.0026	0.2896	0.2777	0.3725	Mle
	2.0500	1.6862	0.1640	0.3262	0.0045	0.4635	0.3699	0.5685	Mle
60	0.8500	0.4005	0.2049	0.2755	0.1314	0.0077	0.0031	0.0316	Mle
	1.1500	0.7219	0.2096	0.1064	0.0499	0.0525	0.0758	0.1039	Mom
	1.4500	1.0434	0.2142	0.0628	0.0317	0.1375	0.1923	0.2174	Mom
	1.7500	1.3648	0.2189	0.2319	0.1133	0.2626	0.2567	0.3711	Mle
	2.0500	1.6862	0.2236	0.4011	0.1949	0.4279	0.5656	0.3303	reg
80	0.8500	0.4005	0.2008	0.4318	0.6643	0.0080	0.0002	0.0314	Mle
	1.1500	0.7219	0.2053	0.1770	0.2520	0.1035	0.0594	0.0534	reg
	1.4500	1.0434	0.2099	0.0778	0.1603	0.1389	0.1865	0.2172	Mom
	1.7500	1.3648	0.2145	0.3326	0.5727	0.2647	0.2131	0.3708	Mle
	2.0500	1.6862	0.2190	0.5874	0.9850	0.4305	0.2415	0.5648	Mle

Table 3.3: Estimator fuzzy hazard rate when (c=5, b=3.2, ʒ=0.8, $\breve{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.0078	0.1272	0.0406	0.0053	0.0068	0.0004	0.0001	Reg
	1.1500	0.1484	0.1281	0.0341	0.0044	0.0026	0.0004	0.0044	Mle
	1.4500	0.2891	0.1290	0.0275	0.0036	0.0137	0.0051	0.0167	Mle
	1.7500	0.4297	0.1299	0.0209	0.0027	0.0180	0.0334	0.0369	Mom
	2.0500	0.5703	0.1308	0.0143	0.0018	0.0386	0.0618	0.0650	Mom
60	0.8500	0.0078	0.1773	0.0542	0.3271	0.0057	0.0002	0.0001	Reg
	1.1500	0.1484	0.1791	0.0455	0.2731	0.0002	0.0021	0.0041	Mom
	1.4500	0.2891	0.1810	0.0367	0.2191	0.0127	0.0023	0.0163	Mle
	1.7500	0.4297	0.1828	0.0280	0.1651	0.0122	0.0323	0.0365	Mom
	2.0500	0.5703	0.1846	0.0192	0.1110	0.0607	0.0297	0.0649	Mle
80	0.8500	0.0078	0.1928	0.0697	0.7447	0.0023	0.0002	0.0001	Reg
	1.1500	0.1484	0.1951	0.0584	0.6217	0.0001	0.0016	0.0040	Mom
	1.4500	0.2891	0.1974	0.0471	0.4987	0.0117	0.0017	0.0160	Mle
	1.7500	0.4297	0.1997	0.0358	0.3758	0.0106	0.0310	0.0362	Mom
	2.0500	0.5703	0.2020	0.0245	0.2528	0.0596	0.0271	0.0645	Mle

Table 3.4: Estimator fuzzy hazard rate when (c=5, b=3.2, ʒ=0.8, $\breve{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.4063	0.1302	0.0109	0.0058	0.0313	0.0152	0.0391	Mle
	1.1500	0.6875	0.1319	0.0043	0.0022	0.0617	0.0934	0.0945	Mom
	1.4500	0.9688	0.1337	0.0024	0.0014	0.1877	0.1868	0.1395	reg
	1.7500	1.2500	0.1355	0.0090	0.0050	0.3100	0.3080	0.2484	reg
	2.0500	1.5312	0.1373	0.0156	0.0086	0.3886	0.4594	0.4688	Mom
60	0.8500	0.4063	0.1625	0.0546	0.1817	0.0119	0.0247	0.0386	Mom
	1.1500	0.6875	0.1654	0.0220	0.0689	0.0886	0.0545	0.0939	Mle
	1.4500	0.9688	0.1683	0.0105	0.0439	0.1836	0.1281	0.1872	Mle
	1.7500	1.2500	0.1712	0.0431	0.1566	0.2328	0.2913	0.3097	Mom
	2.0500	1.5312	0.1741	0.0756	0.2694	0.4238	0.3684	0.4637	Mle
80	0.8500	0.4063	0.1952	0.1952	0.0249	0.0110	0.0089	0.0295	Mle
	1.1500	0.6875	0.1995	0.1995	0.0106	0.0538	0.0476	0.0917	Mle
	1.4500	0.9688	0.2038	0.2038	0.0038	0.0442	0.1170	0.1862	Mom
	1.7500	1.2500	0.2082	0.2082	0.0182	0.3035	0.2171	0.1579	reg
	2.0500	1.5312	0.2125	0.2125	0.0326	0.2716	0.3478	0.4292	Mom

Table 4: Includes Table 4.1, Table 4.2, Table 4.3 and Table 4.4

Table 4.1: Estimator fuzzy hazard rate when (c=3, b=2.8, ʒ=0.8, $\check{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.0329	0.1002	0.4426	0.1682	0.0055	0.0945	0.0081	Mom
	1.1500	0.0635	0.1007	0.3708	0.1404	0.0010	0.0589	0.0012	Mom
	1.4500	0.1599	0.1013	0.2989	0.1127	0.0007	0.0139	0.0004	Reg
	1.7500	0.2564	0.1018	0.2271	0.0849	0.0048	0.0020	0.0059	Mom
	2.0500	0.3528	0.1023	0.1553	0.0571	0.0125	0.0078	0.0275	Mle
60	0.8500	0.0329	0.1134	0.6543	0.0919	0.0043	0.0423	0.0031	Reg
	1.1500	0.0635	0.1142	0.5501	0.0768	0.0005	0.0474	0.0001	Reg
	1.4500	0.1599	0.1149	0.4459	0.0616	0.0004	0.0133	0.0019	Mom
	1.7500	0.2564	0.1156	0.3416	0.0464	0.0040	0.0015	0.0045	Mle
	2.0500	0.3528	0.1163	0.2374	0.0312	0.0112	0.0027	0.0227	Mle
80	0.8500	0.0329	0.1337	0.4942	0.0487	0.0035	0.0356	0.0013	Reg
	1.1500	0.0635	0.1347	0.4157	0.0406	0.0003	0.0248	0.0001	Mom
	1.4500	0.1599	0.1358	0.3373	0.0326	0.0001	0.0123	0.0002	Mom
	1.7500	0.2564	0.1368	0.2588	0.0246	0.0029	0.0010	0.0032	Mle
	2.0500	0.3528	0.1379	0.1804	0.0165	0.0092	0.0017	0.0206	Mle

Table 4.2: Estimator fuzzy hazard rate when (c=3, b=2.8, ʒ=0.8, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.2403	0.1454	0.1952	0.0296	0.0033	0.0026	0.0089	Mle
	1.1500	0.4332	0.1478	0.0891	0.0112	0.0205	0.0237	0.0356	Mom
	1.4500	0.6260	0.1502	0.0169	0.0072	0.0553	0.0742	0.0766	Mom
	1.7500	0.8189	0.1526	0.1230	0.0255	0.0988	0.0969	0.1259	Mle
	2.0500	1.0117	0.1550	0.2290	0.0439	0.1468	0.1225	0.1873	Mle
60	0.8500	0.2403	0.1115	0.3508	0.0302	0.0024	0.0022	0.0088	Mle
	1.1500	0.4332	0.1128	0.1417	0.0114	0.0163	0.0170	0.0356	Mom
	1.4500	0.6260	0.1141	0.0675	0.0073	0.0524	0.0634	0.0761	Mom
	1.7500	0.8189	0.1155	0.2767	0.0260	0.0981	0.0588	0.1257	Mle
	2.0500	1.0117	0.1168	0.4858	0.0447	0.1402	0.0553	0.1870	Mle
80	0.8500	0.2403	0.1300	0.1263	0.0214	0.0018	0.0004	0.0078	Mle
	1.1500	0.4332	0.1319	0.0521	0.0081	0.0157	0.0167	0.0351	Mom
	1.4500	0.6260	0.1338	0.0222	0.0052	0.0485	0.0629	0.0756	Mom
	1.7500	0.8189	0.1356	0.0965	0.0185	0.0934	0.0444	0.1243	Mle
	2.0500	1.0117	0.1375	0.1708	0.0318	0.1399	0.0414	0.1796	Mle

Table 4.3: Estimator fuzzy hazard rate when (c=3, b=3.2, ʒ=0.8, $\check{k}$=0.3)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.0047	0.0812	0.4008	0.1008	0.0025	0.0614	0.0018	Reg
	1.1500	0.0891	0.0816	0.3364	0.0842	0.0001	0.0164	0.0010	Mom
	1.4500	0.1734	0.0819	0.2720	0.0675	0.0017	0.0077	0.0054	Mom
	1.7500	0.2578	0.0823	0.2076	0.0509	0.0062	0.0005	0.0086	Mle
	2.0500	0.3422	0.0826	0.1432	0.0342	0.0135	0.0079	0.0190	Mle
60	0.8500	0.0047	0.1038	0.5352	0.0244	0.0020	0.0563	0.0008	Reg
	1.1500	0.0891	0.1044	0.4491	0.0204	0.0001	0.0159	0.0009	Mom
	1.4500	0.1734	0.1050	0.3631	0.0164	0.0009	0.0072	0.0049	Mom
	1.7500	0.2578	0.1056	0.2770	0.0123	0.0046	0.0003	0.0021	Mle
	2.0500	0.3422	0.1062	0.1909	0.0083	0.0111	0.0046	0.0123	Mle
80	0.8500	0.0047	0.1088	0.0928	0.0581	0.0022	0.0016	0.0006	Reg
	1.1500	0.0891	0.1094	0.0783	0.0485	0.0001	0.0022	0.0003	Mom
	1.4500	0.1734	0.1101	0.0639	0.0389	0.0008	0.0024	0.0036	Mom
	1.7500	0.2578	0.1108	0.0494	0.0293	0.0043	0.0001	0.0004	Mle
	2.0500	0.3422	0.1115	0.0349	0.0197	0.0106	0.0035	0.0108	Mle

Table 4.4: Estimator fuzzy hazard rate when (c=3, b=3.2, ʒ=0.8, $\check{k}$=0.6)

n	t_i	h(t)	$\hat{h}_{mom}$	$\hat{h}_{mle}$	$\hat{h}_{reg}$	Mse-mom	Mse-mle	mse-reg	Best
40	0.8500	0.2438	0.0836	0.1230	0.0589	0.0051	0.0029	0.0068	Mle
	1.1500	0.4125	0.0843	0.0517	0.0223	0.0215	0.0260	0.0304	Mom
	1.4500	0.5813	0.0851	0.0197	0.0142	0.0492	0.0631	0.0643	Mom
	1.7500	0.7500	0.0858	0.0911	0.0507	0.0882	0.0887	0.0978	Mom
	2.0500	0.9187	0.0865	0.1624	0.0873	0.1385	0.1144	0.1383	Mle
60	0.8500	0.2438	0.0964	0.1236	0.0735	0.0043	0.0029	0.0058	Mle
	1.1500	0.4125	0.0974	0.0528	0.0279	0.0199	0.0259	0.0296	Mom
	1.4500	0.5813	0.0984	0.0180	0.0177	0.0630	0.0466	0.0635	Mle
	1.7500	0.7500	0.0994	0.0889	0.0634	0.0847	0.0874	0.0943	Mom
	2.0500	0.9187	0.1004	0.1597	0.1090	0.1339	0.1132	0.1311	Mle
80	0.8500	0.2438	0.1117	0.1243	0.0545	0.0035	0.0029	0.0046	Mle
	1.1500	0.4125	0.1131	0.0614	0.0207	0.0179	0.0247	0.0286	Mom
	1.4500	0.5813	0.1144	0.0014	0.0132	0.0436	0.0626	0.0631	Mom
	1.7500	0.7500	0.1158	0.0643	0.0470	0.0804	0.0868	0.0934	Mom
	2.0500	0.9187	0.1172	0.1271	0.0808	0.1285	0.1123	0.1304	Mle

Conclusion

Table 1 includes four sub Tables due to different sets of initial values of parameters $(b,\ c, \delta, \check{k})$ and all the results of estimation were compared using Mean Square Error (MSE). We found that MOM, is best for all inputs of Table 1, with a percentage of 51%, while MLE estimators of fuzzy hazard rate function was preferred with 28% and the third fuzzy hazard estimator was obtained from regression estimation with 21%.

From Table 2 and according to the different results for estimating $\hat{h}\ (t_i)$, we concluded $\hat{h}\ (t_i)_{mom}$ was best with a percentage of 38%, the $\hat{h}\ (t_i)_{mle}$ was best with 48%, and the third one $\hat{h}\ (t_i)_{reg}$ which was best with 14%.

For Table 3 and according to different results for estimating fuzzy hazard rate function $\hat{h}\ (t_i)$, by three different methods, we concluded the $\hat{h}\ (t_i)_{mom}$ was best with 36%, $\hat{h}\ (t_i)_{mle}$ was best with45%, and $\hat{h}\ (t_i)_{reg}$ was best with 19%.

And finally for Table 4 the results indicated that $\hat{h}\ (t_i)_{mom}$ was best with 43% , $\hat{h}\ (t_i)_{mle}$ was best with 43% and $\hat{h}\ (t_i)_{reg}$ was best with 14%.

The best estimators of $\hat{h}\ (t_i)$ was $\hat{h}\ (t_i)_{mle}$, between all alternatives due to its excellent properties that were satisfied by MLE estimators.

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