Numerical Solution for Nonlinear High-Order Volterra and Fredholm Differential Equation Using Boubaker Polynomial Method

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In this research, the Boubaker polynomials method is utilised to find an approximate solution for the initial value problem of nonlinear high order to Volterra and Fredholm integro differential equation of the second kind. The solution technique is to transform the Volterra and Fredholm nonlinear integro differential equation to a system of nonlinear equations. The proposed method was applied for three different examples and the simulation results were illustrated in tables and graphics. The results were also compared with the exact solution and other's work.

Key words: *Approximation solutions, Boubaker polynomials, Mixed Volterra - Fredholm Integro.*

Introduction

Integro and integral differential equations played an essential role in depicting numerous physical, biological, social and engineering problems. Nonlinear integro and integral differential equations are difficult to solve by analytical methods. Therefore, researchers rarely dealt with Nonlinear Volterra_Fredholm Integro Differential Equations (NVFIDE's) for the difficulty of solving them. However, many of them dealt with linear integral differential equations such as Davaeifarl, *et al.* (2016), who used the Bernstein technique for resolving linear mixed Fredholm differential-difference equations.

Several algorithms appeared for solving a nonlinear integro-differential equations. Yaslanand and Dascioglu developed a method for a Chebyshev collocation to solve nonlinear Fredholm-

Volterra integro-differential equations by transforming the nonlinear integro-differential equation into a matrix equation (Yaslan and Dascioglu, 2006). Biazar and Eslami applied Homotopy Perturbation method to get a solution for the nonlinear mixed Fredholm-Volterra integro equations (Biazar and Eslami, 2010). Behiry used Adomian polynomials with the differential transform method to determined the solution for nonlinear integro-differential equations (Behiry Salah, 2013). Henryk *et al.* developed polynomials techniques such as Taylor polynomials to solve the same problem (Henryk Maleknejad and Mahmoudi, 2003). On the other hand, some researchers deal with Numerical solutions by using mathematical models (Ioan and Egri, 2006). Behrooz et al. used the derivative of the Bernstein operational matrix with initial conditions (Behrooz and Mohammad, 2013). The Boubaker Polynomials extension has been utilised by numerous applied physicists and engineering researchers. Agida and Kumar (2010) utilised this procedure to create an analytical method for resolving integral equation with a rational kernel. Sara and Jalil (2017) used the Boubaker Polynomials method and collocations points for resolving a system of nonlinear mixed Volterra – Fredholm integral equations of the second kind by utilising the classical operational matrices derived. Tinggang Zhao and Yongjun Li (2007) presented a spectral method by using Boubaker Polynomials to resolve differential equations.

The approach concerned transforming the prime problem into an equivalent equation and then replacing the integral parts of this equation by its approximation solutions of the Boubaker polynomials method. This is consequently converted to a system of nonlinear algebraic equations by applying the collocation points x_j. In the next plan, the collocation points are only used with the purpose of decreasing the resulting equation into a system of nonlinear algebraic equations by approximating the unknown functions and the unknown nonlinear coefficients underneath the integrals symbols.

The chief characteristics of the Boubaker Polynomials collocation technique over their comparative methods are in the uncomplicatedness of application together with the legality and trustworthiness of the obtained results.

The major focus of the research is on the NVFIDE's. The nonlinear of the second kind to Volterra and Fredholm differential equation form (Jerri, 1999), (Wazwaz, 2011), (Behiry Sa1ah and Mohamed Saied, 2012):

$$y(x) = g(x) + \delta_1 \int_a^x \lambda_1(x,t)\,[\,y(t)\,]^r\,dt + \delta_2 \int_a^b \lambda_2(x,t)\,[\,y(t)\,]^s\,dt \qquad \ldots (1)$$

where $\lambda_1(x, t)$, $\lambda_2(x, t)$ and $g(x)$ are known functions, δ_1, δ_2, a, b are constant va1ues, r, s are integer numbers and $y(x)$ is the unknown function must be resolved.

Therefor the high order NVFIDEs of the second kind can be written in general form as (Behiry Sa1ah and Mohamed Saied, 2012):

$$\sum_{k=0}^{m} \mu_k \text{ (x) } [y^{(k)} \text{ (x)}] = g\text{ (x)} + \int_a^x \lambda_1 (x, t) [\, y(t) \,]^r \, dt + \int_a^b \lambda_2 (x, t) [\, y(t) \,]^s \, dt \qquad ...(2)$$

With the initial conditions

$y^k (a) = y_k$, $k = 0, 1, 2, . . . , m - 1$.

The goal of this research is to afford numerical technique to approximate the solution y(x) of equation (1) by the Boubaker polynomials. To this end the main focus of the work was nonlinear high order to Volterra and Fredholm integro differential equation of the second kind.

Approximation technique is introduced in this paper to solve the high order NVFIDEs of the second kind utilising Boubaker polynomials technique. The research is displayed as follows: Boubaker polynomials were shown in the second part of this paper. The numerical approximation method and illustrated examples are shown in the third and fourth parts respectively.

Boubaker Polynomials Technique

The Boubaker polynomials B_m (t) of degree m is defined by Boubaker (2007), Yücel and Boubaker (2010), Dada et al. (2011), Yalçınbaş, and Tuğçe (2012) as:

$$B_m(t) = \sum_{q=0}^{\xi(m)} \left[\frac{(m-4q)}{(m-q)} c_{m-q}^{q} \right] (-1)^{\mathbf{q}} \, \mathbf{t}^{m-2q} \qquad ... \; (3)$$

Where, $\xi(m) = \left\lfloor \frac{m}{2} \right\rfloor = \frac{2m + ((-1)^m - 1)}{4}$, where $\xi(m) = \lfloor \; \rfloor$ denotes the floor function.

Therefore, the following characteristics are satisfied (Milovanovic and Joksimovic, 2013):

(i) $B_m(0) = 2\cos\left(\frac{n+2}{2}\pi\right)$, $n \geq 1$

(ii) Boubaker polynomials are even and odd functions,

$B_m(-t) = (-1)^n \, B_m(t)$ $n \in N$.

Thus the standard Boubaker polynomials are defined as (Boubaker, 2007) , (Dada et al., 2011);

$B_0(t) = 1$

$B_1(t) = t$
$B_2(t) = t^2 + 2$
$B_3(t) = t^3 + t$ . . . (4)
:
$B_m(t) = t\, B_{m-1}(t) - B_{m-2}(t)$ for $m > 2$

Boubaker Polynomials to Approximate Functions

In this section, the Boubaker polynomials approximation solution will be considered as a series of finite terms

$$y(t) = \sum_{m=0}^{M} c_m B_m(\mathbf{t}) \qquad -\infty < a \le t \le b < \infty \qquad . . . (5)$$

$B_n(t)$ and c_m for $m = 0, 1, 2, \ldots, M$; $0 \le m \le M$ denote the Boubaker polynomials terms and unknown Boubaker polynomials coefficients respectively.

In finding the approximate solution in the formula of equation (5), which achieves equation (2), by using expanded polynomials Boubaker in Equation (5) and substitute into equation (2) we get:

$$\sum_{i=0}^{n} \mu_i(x) \left[\sum_{m=0}^{M} c_m B_m(x)\right]^i = g(x) + \int_a^x k_1(x,t) \left[\sum_{m=0}^{M} c_m B_m(\mathbf{t})\right]^r dt$$
$$+ \int_a^b k_2(x,t) \left[\sum_{m=0}^{M} c_m B_m(\mathbf{t})\right]^s dt \qquad . . . (6)$$

Equation (6) can be expressed in easy way:

$$\sum_{i=0}^{n} \mu_i(x) \; [c_0 . B_0(x) + c_1 . B_1(x) + c_2 . B_2(x) + c_3 . B_3(x) + c_4 . B_4(x) + \ldots]^i = g(x)$$
$$+ \int_a^x k_1(x,t) \; [c_0 . B_0(x) + c_1 . B_1(x) + c_2 . B_2(x) + c_3 . B_3(x) + c_4 . B_4(x) + \ldots]^r dt \qquad . . . (7)$$
$$+ \int_a^b k_2(x,t) \; [c_0 . B_0(x) + c_1 . B_1(x) + c_2 . B_2(x) + c_3 . B_3(x) + c_4 . B_4(x) + \ldots]^s dt$$

Sequentially, the repetition relation of polynomials terms as equation (4) substitute into equation (7) gives:

$$\sum_{i=0}^{n}\mu_i\,(x)\ [\,c_0 + c_1 . x + c_2 . (x^2+2) + c_3 . (x^3+x) + c_4 .\ (x^4-2)\ + . . .\,]^i = g\,(x)$$
$$+\int_a^x k_1(x,t)\,[c_0 + c_1 . x + c_2 . (x^2+2) + c_3 . (x^3+x) + c_4 .\ (x^4-2)\ + . . .]^r\,dt \qquad . . . (8)$$
$$+\int_a^b k_2(x,t)\,[c_0 + c_1 . x + c_2 . (x^2+2) + c_3 . (x^3+x) + c_4 . (x^4-2)\ + . . .]^s\,dt$$

Next, derivation and integration of all term of Boubaker polynomials at the left side and in the right side of equation (8) respectively, with simplify the above equation. The resulting equation is illustrated as nonlinear equations that involve x as a variable.

The final step is to use the collocation points which are defined as:

$$L = \frac{b-a}{h}$$
$$x_j = a + j\,L\ ;\ \ j = 0, 1, 2, . . ., h\ . \qquad . . . (9)$$

By substituting x_j collocation points in interval $a \le x \le b$, which can be computed from equation (9), and with the initial conditions into equation (8), the set of linear equations, including (m + 1) unknown coefficients c_j , $j = 0, 1, 2, . . . , m$, are established. The coefficients are calculated through solving the system of linear equations utilising MATLAB program (Xue and Chen, 2011).

Numerical Examples

In this part, the numerical method NVFIDE's considered in this research is used to find numerical solutions of three numerical experiments (Biazar and Eslami, 2010), (Behiry Salah, 2013), (Wazwaz, 2011). In addition, the MATLAB function fsolve has been utilised to resolve the current system, so the application of this function has been helpful in finding a correct numerical solution of the system for all examples.

Example 1:

Consider the second kind of integro differential equation for Volterra and Fredholm:

$$y^{(3)}(x) + y(x) = -\frac{1}{5}x^5 + \frac{2}{3}\,x^3 + \frac{5}{6}\,x^2 + \frac{113}{105}\,x - 1 + \int_0^x y^2(t)\,dt\ + \int_0^1 x\,t\,(x+t)\,y^2(t)\,dt$$

Together with y (0) = -1 , y′ (0) = 0 and y″ (0) = 2 as initial conditions and the interval solution $0 \leq x \leq 1$, the exact solution is $y(x) = -1 + x^2$.

Applying the Boubeker polynomials method with a chosen degree of N:

Firstly, let the approximate solution to Example 1 be

$$y(t) = \sum_{j=0}^{N} c_j B_j(t) \text{ , and with initial conditions as } y(0) = \sum_{j=0}^{N} c_j B_j(\mathbf{0}) = -1 ,$$

$$y'(0) = \sum_{j=0}^{N} c_j B'_j(\mathbf{0}) = 0 , \quad y''(0) = \sum_{j=0}^{N} c_j B''_j(\mathbf{0}) = 2 .$$

Now, when N= 1 thus the problem become:

$$\sum_{j=0}^{1} c_j B''_j(\mathbf{x}) + \sum_{j=0}^{1} c_j B_j(\mathbf{x}) = -\frac{1}{5}x^5 + \frac{2}{3}x^3 + \frac{5}{6}x^2 + \frac{113}{105}x - 1$$
$$+ \int_0^x \left[\sum_{j=0}^{1} c_j B_j(\mathbf{t})\right]^2 dt + \int_0^1 x\,t\,(x+t)\left[\sum_{j=0}^{1} c_j B_j(\mathbf{t})\right]^2 dt$$

Where $y(x) = \sum_{j=0}^{1} c_j B_j(x) = c_0 B_0(x) + c_1 B_1(x)$ and by using the standard Boubeker polynomials description in equation (4) yield $y(x) = \sum_{j=0}^{1} c_j B_j(x) = c_0 + c_1 x$

By substituting the above terms into example 1 we get

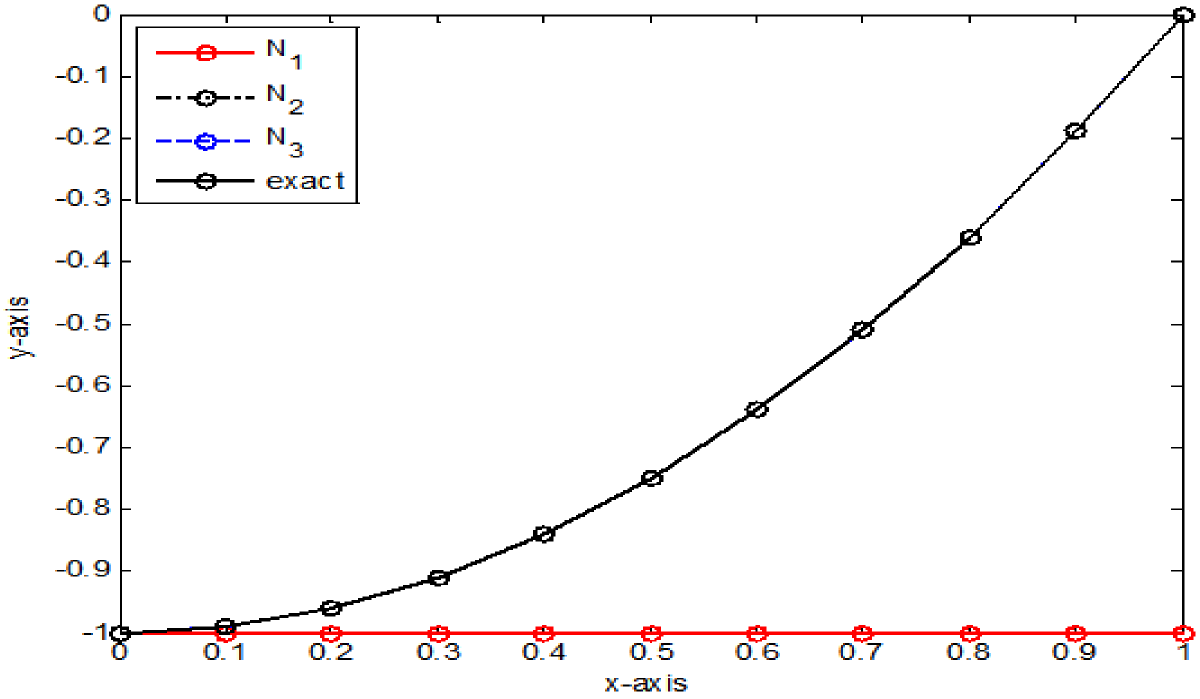
$$c_0 + c_1 x = -\frac{1}{5}x^5 + \frac{2}{3}x^3 + \frac{5}{6}x^2 + \frac{113}{105}x - 1$$
$$+ \int_0^x [c_0 + c_1 t]^2 dt + \int_0^1 x\,t\,(x+t)\,[c_0 + c_1 t]^2 dt$$

Next step is simplifying the above equation and converting the differential integral equation into an algebraic equation, and then selecting some collocation points in the given interval $x \in [0 , 1]$ where $x_j = 0 + 1.j$; j= 0, 1, with initial conditions. The final step of the system of nonlinear equations appears by solving the system by MATLAB code to obtain Boubeker coefficients as c_0, c_1, which are compensated in equation (5) to yield y(x). Through the numerical solutions for N=2 and N=3, we get it by repeating the previous steps. So Tab1e 1 and Figure (1) show the comparison between the exact solutions and with the other work such as differential transform method, and also with our approximation results for various degree N = 1, 2, 3.

Tab1e 1: Comparison between the exact and the approximate solution with differential transform method for various N = 1, 2, 3 and absolute error for Example 1

X	**Exact Solution**	**Differential Transform Method**	**Boubaker polynomials method**			**Absolute error** $\|y(x)_{Exact} - y(x)_{approx.}\|$ N = 3
			N = 1	N = 2	N = 3	
0	- 1. 00000	- 1. 00000	- 1. 00000	- 1. 00000	1. 00000	0.0000
0.1	- 0. 9900	- 0. 99000	- 1. 00000	- 0. 99000	- 0. 99000	0.0000
0.2	- 0. 9600	- 0. 96000	- 1. 00000	- 0. 96000	- 0. 96000	0.0000
0.3	- 0. 9100	- 0. 91000	- 1. 00000	- 0. 91000	- 0. 91000	0.0000
0.4	- 0. 8400	- 0. 84000	- 1. 00000	- 0. 84000	- 0. 84000	0.0000
0.5	- 0. 7500	- 0. 75000	- 1. 00000	- 0. 75000	- 0. 75000	0.0000
0.6	- 0. 6400	- 0. 64000	- 1. 00000	- 0. 64000	- 0. 64000	0.0000
0.7	- 0. 5100	- 0. 51000	- 1. 00000	- 0. 51000	- 0. 51000	0.0000
0.8	- 0. 3600	- 0 .36000	- 1. 00000	- 0. 36000	- 0. 36000	0.0000
0.9	- 0. 1900	- 0. 19000	- 1. 00000	- 0. 19000	- 0. 19000	0.0000
1	0.	0	- 1. 00000	0	0	0.0000

Figure 1. Approximation and exact results for Bubeker degrees N = 1, 2, 3 of example 1.



Examp1e 2:

Solve the following problem

$$x^4 y^{(6)}(x) + y^{(3)}(x) + y'(x) = -x^4 \cos(x) + \frac{1}{2}\sin(2x) + 3x$$

$$+ 0.4 - 0.1 * e * \{ [\cos(1) + \sin(1)] * [\cos^2(1) + 3*e] \} - 2\int_0^x [1 + y^2(t)]\, dt + \int_0^1 e^t y^3(t)\, dt$$

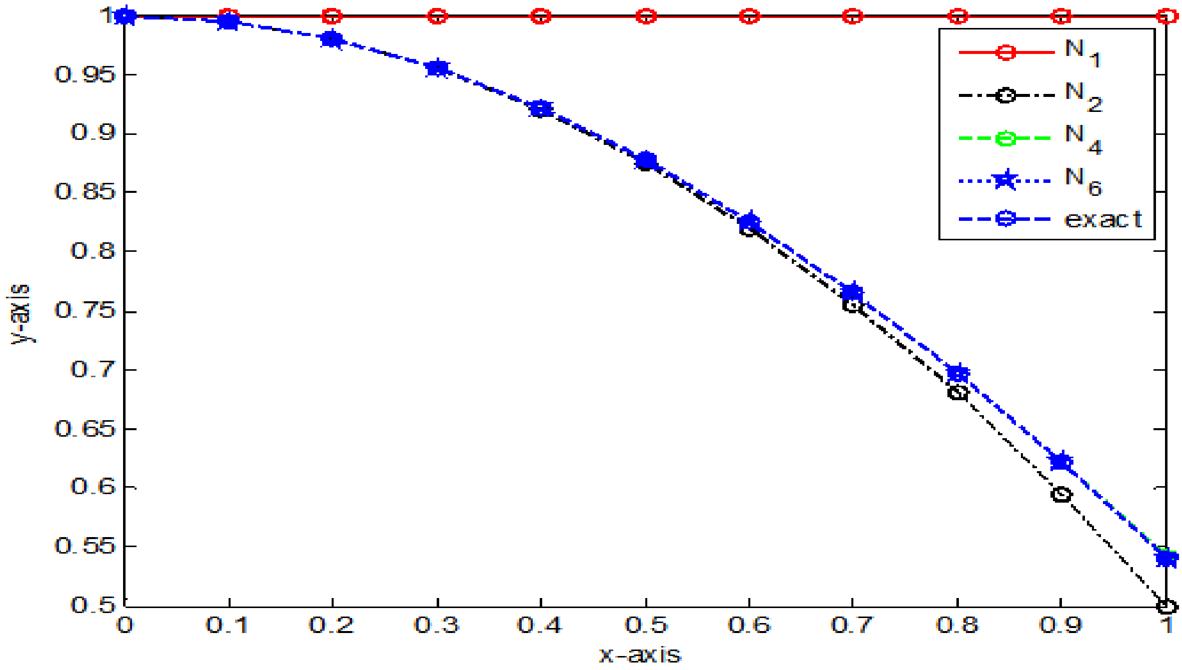
Together with exact solution y (x) = cos (x) and initial conditions: $y(0) = 1$, $y'(0) = 0$, $y''(0) = -1$, $y'''(0) = 0$, $y^{(4)}(0) = 0$ and $y^{(5)}(0) = 0$, $0 \le x \le 1$.

The numerical results in this problem is shown in Table 2 and Figure (2) for different Boubaker degree N = 1 , 2 , 4, 6. Also, compare this with other work such as the transform method.

Tab1e 2: Comparison between the exact and the approximate solution with differential transform method for various N = 1, 2, 4, 6 for Example 2

X	Exact solution	Differential transform method	Boubaker polynomials method			
			N = 1	N = 2	N = 4	N = 6
0	1. 0000	1. 0000	1. 00000	1. 00000	1. 0000	1. 0000
0.1	0. 9950	0. 9950	1.00000	0. 9950	0. 9950	0. 9950
0.2	0. 9801	0. 9801	1.00000	0. 9800	0. 9801	0. 9801
0.3	0. 9553	0. 9553	1.00000	0. 9550	0. 9553	0. 9553
0.4	0. 9211	0. 9211	1. 00000	0. 9200	0. 9211	0. 9211
0.5	0. 8776	0. 8776	1.00000	0. 8750	0. 8776	0. 8776
0.6	0. 8253	0. 8253	1. 00000	0. 8200	0. 8254	0. 8253
0.7	0. 7648	0. 7648	1.00000	0. 7550	0. 7650	0. 7648
0.8	0. 6967	0. 6967	1.00000	0. 6800	0. 6971	0. 6967
0.9	0. 6216	0. 6216	1.00000	0. 5950	0. 6223	0. 6216
1	0. 5403	0. 5403	1.00000	0. 5000	0. 5417	0. 5403

Figure 2. Approximation and exact results for Bubeker degrees N = 1 , 2 , 4, 6 of example 2



Examp1e 3:

Consider the problem as the following

$$y^{(8)}(x) - \pi^8 y(x) = \frac{1}{2}x - \int_0^x \left[y^2(t)\right] dt + dt \int_0^1 \left[\cos(\pi t) - y(t)\right] \frac{\sin(2\pi x)}{2p}$$

With the interval of solution and all initial conditions:

$y(0) = 1$, $y'(0) = \pi$, $y''(0) = 0$, $y'''(0) = -(\pi)^3$, $y^{(4)}(0) = 0$ and $y^{(4)}(0) = 0$, $y^{(5)}(0) = (\pi)^5$, $y^{(6)}(0) = 0$, and $y^{(7)}(0) = -(\pi)^7$, $0 \le x \le 1$

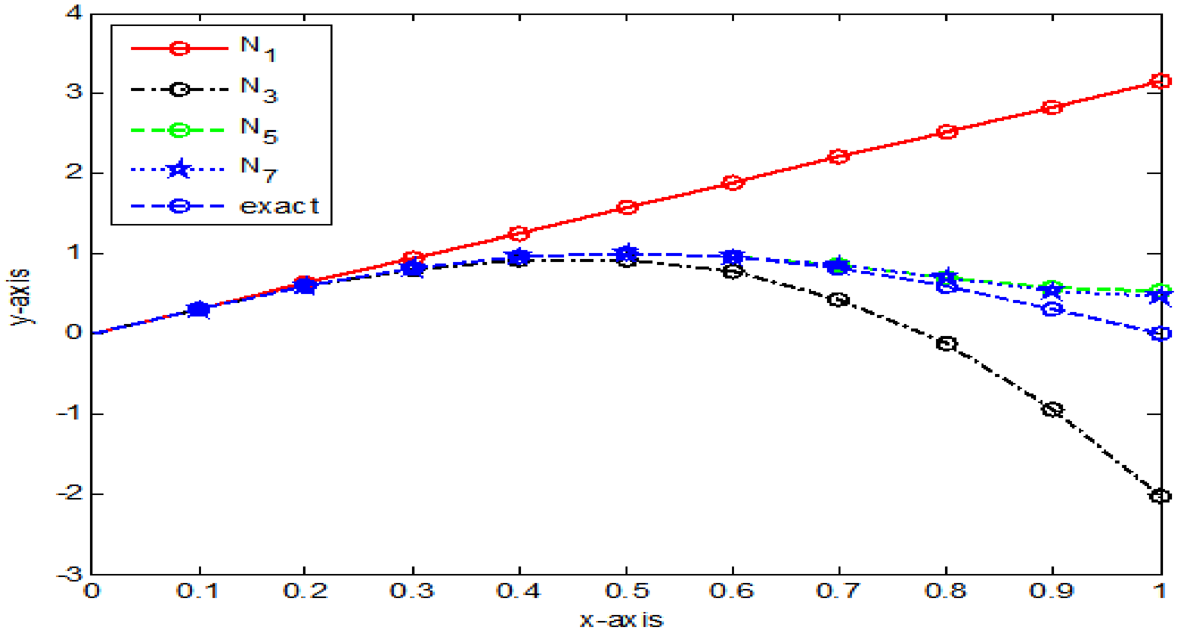
Which is the exact solution $y(x) = \pi x - \frac{\pi^3}{3!} x^3 + \frac{\pi^5}{5!} x^5 - \frac{\pi^7}{7!} x^7$

We can see the numerical results of this problem are shown in Table 3 and Figure (3).

Tab1e 3: Comparison between the exact and the approximate solution with differential transform method for various N = 1, 3, 5, 7 for Example 3

X	Exact so1ution	Differentia1 transform method	Boubaker polynomials method			
			N = 1	**N = 3**	**N = 5**	**N = 7**
0	0	0	0	0	0	0
0.1	0. 308866	0. 3090	0. 3142	0. 3090	0. 3090	0. 3090
0.2	0. 587527	0. 5878	0. 6283	0. 5870	0. 5878	0. 5878
0.3	0. 808734	0. 8091	0. 9425	0. 8029	0. 8091	0. 8091
0.4	0. 950838	0. 9519	1. 2566	0. 9259	0. 9520	0. 9519
0.5	0. 999843	1. 0041	1. 5708	0. 9248	1. 0045	1. 0041
0.6	0. 950553	0. 9653	1. 8850	0. 7687	0. 9670	0. 9653
0.7	0. 806513	0. 8502	2. 1991	0. 4266	0. 8552	0. 8502
0.8	0. 578445	0. 6903	2. 5133	- 0. 1326	0. 7030	0. 6903
0.9	0. 280886	0. 5370	2. 8274	- 0. 9398	0. 5660	0.5370
1	-0. 07329	0. 4633	3. 1416	- 2.0261	0.5240	0.4633

Figure 3. Approximation and exact results for Bubeker degrees N = 1, 3, 5, 7 of example 3



Conclusion

Most integro and integral differential equations are complicated to calculate with analytical methods. Many types are required to find the numerical solutions. The solution of high order of nonlinear of the second kind to Volterra and Fredholm differential equations was solved numerically. In this research, the technique is established on Boubaker polynomials, which reduced a high order of NVFIDEs to a set of linear algebraic equations that can be straightforwardly solved by computer MATLAB. The results showed that the method used can handle those problems effectively as can be seen in the Tables and Figures. Also, when the order of using the Boubaker po1ynomials are increased the results are near to the exact solution as shown in Figure (1), Figure (2) and Figure (3), and fit with other works such as the transform method as seen in Table 1, Table 2 and Table 3.

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